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On Splitting the Area under Curves

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Two area-splitting problems involving real-valued functions of a real variable are investigated. The second of these is essentially equivalent to finding all functions $a \in C^1((0, r))$ with $0 < a(x) < x$ which satisfy the functional differential equation $a'(a(x)) = a(x)/x$ for $x \in (0, r)$. All solutions analytic at $x = 0$ (and many which are not) are exhibited in closed form.

1. INTRODUCTION

This paper deals with two area problems involving real-valued functions of a real variable. Interestingly, the problems arose through a student's error on a calculus examination. In the first problem (Sect. 2), we find all continuous, strictly monotone functions f such that, on any closed and bounded interval within the domain of f, the area under the graph of f is divided in a certain fixed ratio $\alpha/(1 - \alpha)$ (independently of the choice of interval) at the place where f takes its average value. In the second problem (Sect. 3), we again seek functions with the above area-splitting property, but only on all intervals with left endpoint zero.

We see in Theorem 1 that the first problem has no solution unless $\alpha = \frac{1}{2}$, in which case $f(t) = \pm 1/(c_1 t + c_2)^{1/2}$. The proof is elementary but calculational; perhaps one of its more attractive features is that it contains a simple application (in Lemma 5) of the result that the continuity of mixed partial derivatives implies their equality.

The second problem is more interesting and much harder, for it is related to Schröder's equation, and it is essentially equivalent to finding all functions $a \in C^1((0, r))$ with $0 < a(x) < x$ which satisfy the functional differential equation

$$a'(a(x)) = a(x)/x \qquad \text{for} \quad x \in (0, r). \tag{$*$}$$

The quantity α is related to solutions of $(*)$ via an initial condition. Equations (11) and (12) arising in the first problem will yield a family of solutions (Theorem 17) to the second problem (but with a surprising shift in α), and these in turn yield a family of solutions (27) for $(*)$. Again the proofs are elementary but involve some intricate calculations. Functions with the area-

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splitting property for every α except $1/e$ are found, the case $1/e$ being unresolved. All solutions of $(*)$ with $a(0) = 0$ which are analytic at $x = 0$ (and many which are not) are exhibited in closed form. (See (27) and Theorems 23 and 24.) Section 4 is concerned with the existence and uniqueness of analytic solutions for $(*)$ and for the area-splitting problem. Section 5 contains a list of open questions.

2. The "Wombat" Problem

Fix $0 < \alpha < 1$ and an interval (p, q). Let $f\colon (p, q) \to \mathbb{R}$ be a continuous, strictly monotone function. For all $u, x \in (p, q)$, define

$$F(u, x) = \int_u^x f(t)\, dt. \tag{1}$$

For $p < u < x < q$, let $a(u, x)$ be the unique point in (u, x) where the function f takes its average value on $[u, x]$; that is,

$$f(a(u, x)) = F(u, x)/(x - u). \tag{2}$$

Definition. The function f will be called an α-*wombat function*[1] on (p, q) if

$$F(u, a(u, x)) = \alpha F(u, x) \tag{3}$$

for all $p < u < x < q$.

We prove the following result.

Theorem 1. *The functions*

$$f(t) = \pm 1/(c_1 t + c_2)^{1/2}, \qquad c_1, c_2 \textit{ constants}, \quad c_1 \neq 0, \tag{4}$$

are $\frac{1}{2}$-wombat functions on their entire domains, and every $\frac{1}{2}$-wombat function is of this form. For $\alpha \neq \frac{1}{2}$, no α-wombat functions exist.

The proof proceeds by a series of lemmas. Lemma 4 below is crucial, since it assures us that formal partial differentiation of (3) is justified.

Let f be an α-wombat function on (p, q).

Lemma 2. *If $u < x$, $F(u, x) \neq 0$.*

Proof. If $F(u, x) = 0$, then (2) implies $f(a(u, x)) = 0$. By monotonicity, f is of one sign on $[u, a(u, x)]$ and so $F(u, a(u, x)) \neq 0$, contradicting (3).

[1] In honor of F. Wombat, a sobriquet for the student whose creative incompetence in freshman calculus inspired this research.

LEMMA 3. *f is never zero on (p, q).*

Proof. Suppose $f(t_0) = 0$. Using monotonicity and the intermediate-value theorem (applied to the area function), we could find an interval (u, x) containing t_0 with $[u, x] \subset (p, q)$ and $F(u, x) = 0$, contradicting Lemma 2.

LEMMA 4. *For all $u < x$, $a_u(u, x)$ and $a_x(u, x)$ exist and*

$$a_x(u, x) = \alpha(x - u) f(x)/F(u, x), \tag{5}$$

$$a_u(u, x) = (1 - \alpha)(x - u) f(u)/F(u, x). \tag{6}$$

Proof. From (3),

$$F(u, a(u, x + \Delta x)) = \alpha F(u, x + \Delta x)$$

and

$$F(u, a(u, x)) = \alpha F(u, x).$$

Subtraction yields $F(a(u, x), a(u, x + \Delta x)) = \alpha F(x, x + \Delta x)$. By the mean-value theorem, this becomes

$$((a(u, x + \Delta x) - a(u, x))/\Delta x) \cdot f(\theta_1) = \alpha f(\theta_2), \tag{7}$$

where $\theta_1 \in [a(u, x), a(u, x + \Delta x)]$ and $\theta_2 \in [x, x + \Delta x]$. Since $f(a(u, x)) = F(u, x)/(x - u)$,

$$f(a(u, x + \Delta x)) = F(u, x + \Delta x)/(x + \Delta x - u),$$

and f is monotone, we get

$$f(\theta_1) \in [F(u, x)/(x - u), F(u, x + \Delta x)/(x + \Delta x - u)],$$

so $\lim_{\Delta x \to 0} f(\theta_1) = F(u, x)/(x - u)$, nonzero by Lemma 2. Letting $\Delta x \to 0$ in (7), we get both the existence of $a_x(u, x)$ and Eq. (5). Equation (6) follows analogously from the equation

$$F(a(u, x), x) = (1 - \alpha) F(u, x). \tag{3'}$$

LEMMA 5. *For all $u < x$,*

$$[(1 - \alpha) f(u) + \alpha f(x)] F(u, x) = (x - u) f(u) f(x). \tag{8}$$

Proof. Fix x in (5). We see that $a_{xu}(u, x)$ exists and

$$a_{xu}(u, x) = \alpha f(x)(((x - u) f(u) - F(u, x))/(F(u, x))^2). \tag{9}$$

Similarly, from (6),

$$a_{ux}(u, x) = (1 - \alpha) f(u)((F(u, x) - (x - u) f(x))/(F(u, x))^2). \tag{10}$$

Since (9) and (10) say that a_{xu} and a_{ux} are in fact continuous, we know $a_{xu} = a_{ux}$. Equating (9) and (10) yields (8).

LEMMA 6. *$f \in C^1((p, q))$, and for all $u < x$,*

$$\alpha(x-u)[f(x)]^2 f'(u) = (1-\alpha) f(u)[f(x) - f(u)][(1-\alpha) f(u) + \alpha f(x)], \tag{11}$$

$$(1-\alpha)(x-u)[f(u)]^2 f'(x) = \alpha f(x)[f(x) - f(u)][(1-\alpha) f(u) + \alpha f(x)]. \tag{12}$$

Proof. Solving (8) for $f(u)$, we get

$$f(u)[(1-\alpha) F(u,x) - (x-u) f(x)] = -\alpha f(x) F(u,x).$$

By Lemmas 2 and 3, RHS $\neq 0$, so

$$f(u) = -\alpha f(x) F(u,x)/((1-\alpha) F(u,x) - (x-u) f(x)).$$

Fix x. Then we see that $f \in C^1((p, x))$; since x was arbitrary, conclude $f \in C^1((p, q))$. So we can apply $\partial/\partial u$ to (8) to get

$$(1-\alpha) f'(u) F(u,x) = (x-u) f'(u) f(x) - (1-\alpha) f(u) f(x) + (1-\alpha)[f(u)]^2. \tag{13}$$

Eliminating $F(u, x)$ between (8) and (13) yields (11). Analogously, if we apply $\partial/\partial x$ to (8), we get (12).

Proof of Theorem 1. By monotonicity and Lemma 3, the right-hand sides of (11) and (12) never vanish, so the same is true of the left-hand sides. Dividing (12) by (11) yields

$$\frac{f'(x)}{[f(x)]^3} = \left(\frac{\alpha}{1-\alpha}\right)^2 \cdot \frac{f'(u)}{[f(u)]^3} \qquad \text{for all} \quad u < x. \tag{14}$$

It follows that $f'(u)/[f(u)]^3 = \text{constant}$ on (p, q), and then α must be $\frac{1}{2}$ by (14). The only possible solutions of this separable differential equation are functions of the form (4). Since it is an easy calculation to show that all functions of the form (4) are indeed $\frac{1}{2}$-wombat functions, the proof is complete.

3. THE "SEMIWOMBAT" PROBLEM

Fix $0 < \alpha < 1$ and $r > 0$. Let $f\colon [0, r) \to \mathbb{R}$ be a continuous, strictly monotone function. For all $0 \leqslant x < r$, define

$$F(x) = \int_0^x f(t)\, dt. \tag{15}$$

For $0 < x < r$, let $a(x)$ be the unique point in $(0, x)$ where the function f takes its average value on $[0, x]$; that is,

$$f(a(x)) = F(x)/x. \tag{16}$$

DEFINITION. The function f is called an *α-semiwombat function* on $[0, r)$ if

$$F(a(x)) = \alpha F(x) \tag{17}$$

for all $0 < x < r$.

Remark. Clearly Eq. (17) is the special case $u = 0$ in Eq. (3), with $F(0, x)$ and $a(0, x)$ written simply as $F(x)$ and $a(x)$.

Remark. Equation (17) is recognizable as Schröder's equation,[2] when $a(x)$ and α are supposed known. (See [2, Chap. VI].) Unfortunately, the major difficulty in finding semiwombat functions is to find suitable choices for $a(x)$ and α. (See Theorem 12 below and the comment following it.) Once $a(x)$ and α are determined, $F(x)$ is easily found via Eq. (18) below, without recourse to the usual iterative methods. Therefore, although solutions of the semiwombat problem can be viewed as illustrations of the general theory of Schröder's equation, we do not pursue the connections further here.

Let f be an α-semiwombat function on $[0, r)$.

LEMMA 7. *$F(x) \neq 0$ for $x > 0$.*

Proof. Analogous to the proof of Lemma 2.

LEMMA 8. *$f(x) \neq 0$ for $x > 0$.*

Proof. Suppose $f(x_0) = 0$ for some $x_0 > 0$. First assume $f(0) > 0$. Then monotonicity implies $0 < F(x_0 + \epsilon) < F(x_0)$ for $\epsilon > 0$ but small. Then $0 < F(x_0 + \epsilon)/(x_0 + \epsilon) < F(x_0)/x_0$, so, from (16), $a(x_0) < a(x_0 + \epsilon) < x_0$. This implies $F(a(x_0 + \epsilon)) > F(a(x_0)) = \alpha F(x_0) > \alpha F(x_0 + \epsilon)$, contradicting (17). The case $f(0) < 0$ is similar.

LEMMA 9. *For $x > 0$, $a'(x) = \alpha x f(x)/F(x)$. Hence $a \in C^1((0, r))$.*

Proof. Analogous to the proof of Lemma 4.

LEMMA 10. *If $f(0) \neq 0$, then $a'(0) = \alpha$ and $a \in C^1([0, r))$.*

Proof. Since $0 < a(x) < x$, a is continuous at $x = 0$ (assuming, of course, we let $a(0) = 0$). If $f(0) \neq 0$, then Lemma 9 implies $\lim_{x\to 0} a'(x) = \alpha f(0)/F'(0) = \alpha$. It follows from the mean-value theorem that $a'(0)$ exists and equals α.

[2] The author wishes to thank the referee for pointing this out.

THEOREM 11. *Fix $r_0 \in (0, r)$. Then for all $x \in (0, r)$,*

$$F(x) = F(r_0) \exp \int_{r_0}^{x} \frac{a'(t)}{\alpha t}\, dt, \tag{18}$$

$$f(x) = F(r_0) \frac{a'(x)}{\alpha x} \exp \int_{r_0}^{x} \frac{a'(t)}{\alpha t}\, dt. \tag{19}$$

Proof. Lemma 9 implies that $a'(x)/\alpha x = F'(x)/F(x)$. Integrate both sides and take exponentials to get (18). Differentiate (18) to get (19).

THEOREM 12. *The function $a(x)$ satisfies:*

$$a'(a(x)) = a(x)/x \qquad \textit{for} \quad x \in (0, r). \tag{$*$}$$

Proof. Replace x by $a(x)$ in Lemma 9 and use (16) and (17).

The remarkable feature of $(*)$ is that α is not present. We now explore how solutions of $(*)$ can yield α-semiwombat functions, and how a solution is related to α. Roughly speaking, the value of $a'(0)$ determines two values of α.

THEOREM 13. *The function $a(x)$ also satisfies the following three conditions:*

$$\lim_{x \to 0} \int_{r_0}^{x} \frac{a'(t)}{t}\, dt = -\infty, \tag{20}$$

$$\lim_{x \to 0} \frac{a'(x)}{x} \exp \int_{r_0}^{x} \frac{a'(t)}{\alpha t}\, dt \textit{ exists and is finite}, \tag{21}$$

$$\lim_{x \to 0} \int_{x}^{a(x)} \frac{a'(t)}{t}\, dt = \alpha \log \alpha. \tag{22}$$

Proof. Since $\lim_{x \to 0} F(x) = 0$, condition (20) is a consequence of (18). Since f is continuous at $x = 0$, condition (21) is a consequence of (19). From (17) and (18),

$$\exp \int_{r_0}^{a(x)} \frac{a'(t)}{\alpha t}\, dt = \alpha \exp \int_{r_0}^{x} \frac{a'(t)}{\alpha t}\, dt.$$

Take logarithms and let $x \to 0$ to get (22).

We next prove a somewhat inelegant converse to Theorems 12 and 13.

THEOREM 14. *Suppose the following hold:*

(i) $a \in C^1((0, r))$,

(ii) $0 < a(x) < x$,

(iii) *a satisfies $(*)$,*

(iv) *there exist $r_0 \in (0, r)$ and $\alpha \in (0, 1)$ such that* (20), (21), *and* (22) *hold,*

(v) *the function f defined by Eq.* (19) *is strictly monotone.*

Then f is an α-semiwombat function on $[0, r)$.

Proof. We have assumed enough regularity conditions to ensure the continuity of f. Condition (20) ensures that the antiderivative of f as given by (18) is in fact the area function $\int_0^x f(t)\,dt$. Since a satisfies $(*)$, verifying (16) and (17) reduces to showing $\int_x^{a(x)} (a'(t)/t)\,dt = \alpha \log \alpha$. Again using $(*)$, we see that the left-hand side of this equation has a zero derivative, and so is constant. Condition (22) then yields the desired value for the constant.

Remark. It can be shown that if all of the hypotheses of Theorem 14 are satisfied with the possible exception of (20), then (20) must hold also. There may be other overdeterminations; for example, every solution of $(*)$ known to the author which satisfies $0 < a(x) < x$ is continuously differentiable on $(0, r)$. On the other hand, if we drop all regularity assumptions on a (that is, if we completely disregard the geometric problem which gave rise to $(*)$), we can find quite pathological solutions for $(*)$.

EXAMPLES. The function $a(x) = kx$ satisfies $(*)$, but makes no geometric sense unless $k \in (0, 1)$. The function

$$\begin{aligned} a(x) &= x + 1, & x &\in (0, \tfrac{1}{2}), \\ &= x + \log(x - 1), & x &\in (1, \tfrac{3}{2}), \end{aligned}$$

satisfies $(*)$ for $x \in (0, \frac{1}{2})$. More generally, we can think of $(*)$ as a differential-difference equation; that is, given $0 < p_2 < p_1$, we can take a on $[p_2, p_1]$ such that a is continuous and strictly increasing with $a(p_1) = p_2$, and then use $(*)$ to continue a to $[a(p_2), p_2]$, etc. As a specific example of this idea, the reader may verify that the initial choice $a(x) = \frac{1}{2}x^2$ for $x \in [\frac{1}{2}, 1]$ generates a function $a \in C^1((c, 1))$ (where c is some negative constant) which satisfies $(*)$ for $x \in (0, 1)$, and with $a(0) = c$. (Of course, it is not at all clear how to initially specify a so that this method yield solutions of $(*)$ satisfying $a(x) > 0$ for $x > 0$.) Finally, to avoid contrived examples like the last two, one might be tempted to insist that $a(x)$ be continuous at $x = 0$ with $a(0) = 0$. But consider the following function.

Let $a_0 = 1$, $b_0 = 2$ and inductively define

$$\begin{aligned} a_{n+1} &= 4a_n - 1, & b_{n+1} &= 4b_n + 1, \\ a_{-(n+1)} &= \tfrac{1}{4}a_{-n} + \tfrac{1}{2}\cdot 8^{-(n+1)}, & b_{-(n+1)} &= \tfrac{1}{4}b_{-n} - \tfrac{1}{2}\cdot 8^{-(n+1)}, \end{aligned}$$

for $n = 0, 1, 2, \ldots$. Let $I_j = [a_j, b_j]$ for all integers j. Note that $4I_j \subset \operatorname{int} I_{j+1}$ for all j. Define a function $a(x)$ on $x \geqslant 0$ by

$$\begin{aligned} a(x) &= 4x \qquad \text{if } x \in \bigcup_{j=-\infty}^{\infty} I_j \\ &= \tfrac{1}{4}x \qquad \text{otherwise.} \end{aligned}$$

Then $a(x)$ satisfies $(*)$ for all $x > 0$ and is continuous at $x = 0$, but its points of discontinuity cluster at $x = 0$, and $a'(0)$ does not exist.

THEOREM 15. *Suppose* $0 < a(x) < x$ *and* $a \in C^1((0, r))$. *Suppose further that* $a'(t) = a'(0) + q(t)$, *where* $0 < a'(0) < 1$ *and* $\lim_{t\to 0}(q(t)/t^\delta) = 0$ *for some* $\delta > 0$. *Then condition* (22) *holds if and only if* $a'(0) \log a'(0) = \alpha \log \alpha$, *condition* (21) *holds if and only if* $\alpha \leqslant a'(0)$, *and condition* (20) *always holds. If* $\alpha = a'(0)$, *then the limit in* (21) *is nonzero* (*that is,* $f(0) \neq 0$ *in* (19)). *If* $\alpha < a'(0)$, *then the limit in* (21) *is zero* (*that is,* $f(0) = 0$ *in* (19)).

Proof. The proof is routine, and we omit the details. Essentially, one need only observe that

$$\int_0^{r_0} \frac{q(t)}{t}\, dt = \int_0^{r_0} \frac{q(t)}{t^\delta}\, t^{\delta-1}\, dt$$

and $t^{\delta-1}$ is integrable, so the former integral is finite.

Remark. Had we dropped the hypothesis that $0 < a'(0) < 1$, we still could have shown that condition (22) holds if and only if $0 < a'(0) < 1$ and $a'(0) \log a'(0) = \alpha \log \alpha$. So the cases $a'(0) = 0$ and $a'(0) = 1$ are of no further interest.

Now suppose $a(x)$ satisfies $(*)$ and the hypotheses of Theorem 15. An examination of the graph of $x \log x$ for $0 < x < 1$ shows that, if $a'(0) \neq e^{-1}$, there are two values of α satisfying $a'(0) \log a'(0) = \alpha \log \alpha$, say α_1 and α_2, with $\alpha_1 < e^{-1} < \alpha_2$. If $a'(0) > e^{-1}$, then $\alpha_1 < a'(0)$ and $\alpha_2 = a'(0)$, and Theorems 14 and 15 tell us to expect (19) to yield an α_1-semiwombat with $f(0) = 0$ and an α_2-semiwombat with $f(0) \neq 0$. If $a'(0) < e^{-1}$, then $\alpha_1 = a'(0)$ and $\alpha_2 > a'(0)$, and so we expect (19) to yield an α_1-semiwombat with $f(0) \neq 0$. If we use α_2 in (19), we get an α_2-semiwombat save for the fact that f is not continuous at $x = 0$.

We now turn to the problem of solving $(*)$. We begin by noting that if $a(x)$ satisfies $(*)$ then so does $(1/k)\, a(kx)$, $k > 0$, $0 < x < r/k$. This is not surprising; it corresponds to the geometric fact that if $f(x)$ is an α-semiwombat function on $[0, r)$, then $cf(kx)$, $c \neq 0$, $k > 0$, is an α-semiwombat function on $[0, r/k)$. We refer to this idea as "scaling."

As noted earlier, the function $a(x) = kx$, $0 < k < 1$, satisfies $(*)$ and the hypotheses of Theorem 15, with $a'(0) = k$. If we take $\alpha = k$ in Theorems 14 and 15, Eq. (19) yields $f \equiv$ constant. If $k > e^{-1}$, we can take $\alpha < e^{-1}$ with $\alpha \log \alpha = k \log k$, and (19) yields $f(x) = cx^{(k/\alpha)-1}$, $c \neq 0$. If $k < e^{-1}$, we can take $\alpha > e^{-1}$ with $\alpha \log \alpha = k \log k$, and (19) again yields $f(x) = cx^{(k/\alpha)-1}$, $c \neq 0$, but this time (as expected), f is discontinuous at $x = 0$. We now let $\nu = (k/\alpha) - 1$ and summarize our results.

THEOREM 16. *The function* $f(x) = cx^\nu$, $\nu > 0$, $c \neq 0$, *is a* $(\nu + 1)^{-1-1/\nu}$-*semiwombat function on* $[0, \infty)$. (*If we ignore continuity at* $x = 0$, *the above statement also holds for* $-1 < \nu < 0$.)

Are there other semiwombat functions? The $\frac{1}{2}$-wombat function $f(x) = (x + 1)^{-1/2}$ is certainly a $\frac{1}{2}$-semiwombat function, and yields $a(x) = \frac{1}{4}(x - 2 + 2(x + 1)^{1/2})$ as a nontrivial solution of $(*)$. We now look at Eqs. (11) and (12) of Section 2 for further motivation. If we assume $f(0) = 1$, these equations yield

$$\alpha x f'(x) = (1 - \alpha) f(x)[f(x) - 1][(1 - \alpha) f(x) + \alpha], \qquad x < 0, \quad (23)$$

$$(1 - \alpha)\, x f'(x) = \alpha f(x)[f(x) - 1][\alpha f(x) + 1 - \alpha], \qquad x > 0. \quad (24)$$

Equations (23) and (24) do not "piece together" at $x = 0$ unless $\alpha = \frac{1}{2}$. But perhaps each equation separately will yield a semiwombat function. Let $\beta = (1 - \alpha)/\alpha$ in (24). If we solve the separable differential Eq. (24) and ignore scaling, we get two functions $y = f_{\beta,1}(x)$ and $y = f_{\beta,2}(x)$, one increasing and one decreasing. The complete story is as follows.

THEOREM 17. *For each* $0 < \beta < \infty$, *consider the functions* $g_{\beta,1}$ *and* $g_{\beta,2}$ *defined in* $[1, \infty)$ *and* $(0, 1]$, *respectively, by*

$$g_{\beta,1}(y) = y^{-\beta-1}(y - 1)^\beta(y + \beta), \quad (25)$$

$$g_{\beta,2}(y) = y^{-\beta-1}(1 - y)^\beta(y + \beta). \quad (26)$$

Let $f_{\beta,i} = g_{\beta,i}^{-1}$ *for* $i = 1, 2$. *Then the functions* $y = f_{\beta,i}(x)$, $i = 1, 2$, *are* $(1 + 1/\beta)^{-\beta}$-*semiwombat functions on* $[0, 1)$ *and* $[0, \infty)$, *respectively.*

Proof. We prove the theorem only for $f_{\beta,1}(x)$. Since $g'_{\beta,1}(y) = \beta(\beta + 1) y^{-\beta-2}(y - 1)^{\beta-1}$, we see that $f_{\beta,1}$ is strictly increasing on $[0, 1)$. For $0 < t < 1$, $F_{\beta,1}(t) = \int_0^t y\, dx = \int_1^u y\, g'_{\beta,1}(y)\, dy = (\beta + 1)(1 - 1/u)^\beta$, where $u = f_{\beta,1}(t)$. Then

$$f_{\beta,1}(a_{\beta,1}(t)) = \frac{F_{\beta,1}(t)}{t} = \frac{(\beta + 1)(1 - 1/u)^\beta}{g_{\beta,1}(u)} = \frac{(\beta + 1)\, u}{u + \beta}.$$

But then $F_{\beta,1}(a_{\beta,1}(t)) = (\beta + 1)(1 - (u + \beta)/(\beta + 1)u)^\beta = (1 + 1/\beta)^{-\beta}(\beta + 1)(1 - 1/u)^\beta = (1 + 1/\beta)^{-\beta} F_{\beta,1}(t)$. Done.

We next exhibit the corresponding average-value functions $a_{\beta,i}(x)$, $i = 1, 2$. Note that, from the above computations,

$$\begin{aligned} a_{\beta,1}(t) &= g_{\beta,1}[f_{\beta,1}(a_{\beta,1}(t))] = g_{\beta,1}((\beta + 1)u/(u + \beta)) \\ &= (1 + 1/\beta)^{-\beta-1} \beta^{-1}(1 - 1/u)^\beta(2\beta + 1 + \beta^2/u). \end{aligned}$$

Since $t = g_{\beta,1}(u)$, we can use (25) to simplify this somewhat. We summarize, as follows.

COROLLARY 18. *The functions*

$$a_{\beta,i}(x) = \left(1 + \frac{1}{\beta}\right)^{-\beta-1} x \frac{(2\beta + 1) f_{\beta,i}(x) + \beta^2}{\beta f_{\beta,i}(x) + \beta^2} \tag{27}$$

are the average-value functions for $f_{\beta,i}(x)$, $i = 1, 2$, *and therefore satisfy* $(*)$.

We now wish to invoke the machinery of Theorems 14 and 15 to see if (27) yields other semiwombat functions.

LEMMA 19. *Functions* (27) *satisfy the hypotheses of Theorem* 15.

Proof. By a straightforward but tedious calculation,

$$a'_{\beta,i}(x) = (1 + 1/\beta)^{-\beta-1} \beta^{-1}(f_{\beta,i}(x) + \beta), \qquad i = 1, 2. \tag{28}$$

Note that $a'_{\beta,i}(0) = (1 + 1/\beta)^{-\beta}$. (This also follows from Theorem 17 and Lemma 10.) If we let $q_{\beta,i}(t) = a'_{\beta,i}(t) - a'_{\beta,i}(0)$, then

$$\lim_{t\to 0} \frac{q_{\beta,i}(t)}{t^\delta} = \lim_{u\to 1} \frac{c_0(u-1)}{[g_{\beta,i}(u)]^\delta} = 0 \qquad \text{for} \quad \delta < \frac{1}{\beta}.$$

This completes the proof.

Now since $(1 + 1/\beta)^{-\beta}$ decreases from 1 to e^{-1} as β increases from 0 to ∞, equation $a'_{\beta,i}(0) \log a'_{\beta,i}(0) = \gamma \log \gamma$ has two distinct roots $0 < \gamma_1 < e^{-1} < \gamma_2 < 1$, with $\gamma_1 = (1 + 1/\beta)^{-\beta-1}$, $\gamma_2 = (1 + 1/\beta)^{-\beta} = a'_{\beta,i}(0)$. The second root is of no interest; Eq. (19) tells us that a given function $a(x)$ and a given α can determine only one semiwombat function (ignoring scaling), and we already know what that semiwombat function is, namely $f_{\beta,i}$. It is γ_1 which concerns us. If we use γ_1 for α in (19), Theorems 14 and 15 tell us to expect a $(1 + 1/\beta)^{-\beta-1}$-semiwombat $\tilde{f}_{\beta,i}$ with $\tilde{f}_{\beta,i}(0) = 0$.

THEOREM 20. *For each* $0 < \beta < \infty$, *the functions* $\tilde{f}_{\beta,1}(x) = f_{\beta,1}(x) - 1$ *and* $\tilde{f}_{\beta,2}(x) = 1 - f_{\beta,2}(x)$ *are* $(1 + 1/\beta)^{-\beta-1}$-*semiwombat functions on* $[0, 1)$ *and* $[0, \infty)$, *respectively*.

Proof. By the preceding discussion, one need only substitute γ_1 for α and $a_{\beta,i}(x)$ for $a(x)$ in (19), and check the monotonicity of the resulting function. After integrating in (19) and simplifying, we see that (ignoring scaling) the function $\tilde{f}_{\beta,i}$ is just a lowering of $f_{\beta,i}$ through the origin.

For the sake of symmetry, we restate Theorem 20.

THEOREM 21. *For each* $0 < \beta < \infty$, *consider the functions* $\tilde{g}_{\beta,1}$ *and* $\tilde{g}_{\beta,2}$ *defined in* $[0, \infty)$ *and* $[0, 1)$, *respectively, by*

$$\tilde{g}_{\beta,1}(y) = (y + 1)^{-\beta-1} y^\beta (y + \beta + 1), \tag{29}$$

$$\tilde{g}_{\beta,2}(y) = (1 - y)^{-\beta-1} y^\beta (\beta + 1 - y). \tag{30}$$

Let $\tilde{f}_{\beta,i} = \tilde{g}_{\beta,i}^{-1}$ *for* $i = 1, 2$. *Then* $y = \tilde{f}_{\beta,i}(x)$, $i = 1, 2$, *are* $(1 + 1/\beta)^{-\beta-1}$-*semiwombat functions on* $[0, 1)$ *and* $[0, \infty)$, *respectively.*

We next prove that the functions $f_{\beta,i}$ are the *only* semiwombats which remain semiwombats upon lowering through the origin.

THEOREM 22. *Let* f *be a semiwombat function with* $f(0) \neq 0$ *and suppose* $\tilde{f} = f - b$, b *a nonzero constant, is also a semiwombat function. Then, ignoring scaling,* $f = f_{\beta,i}$ *for some* β, i, *and* $b = f(0)$.

Proof. Suppose f is a γ-semiwombat and $\tilde{f}$ a $\tilde{\gamma}$-semiwombat. Since f and $\tilde{f}$ have the same average-value function a, (22) implies $\gamma \log \gamma = \tilde{\gamma} \log \tilde{\gamma}$. If $\gamma = \tilde{\gamma}$, (19) would imply that f and $\tilde{f}$ are constant multiples of each other, forcing $f \equiv$ constant. So $\gamma \neq \tilde{\gamma}$. If $b \neq f(0)$, we would have $\tilde{f}(0) \neq 0$, and then $\gamma = \tilde{\gamma} = a'(0)$ by Lemma 10. So $b = f(0)$. The three equations $F(a(x)) = \gamma F(x)$, $\tilde{F}(a(x)) = \tilde{\gamma}\tilde{F}(x)$, $\tilde{F}(x) = F(x) - bx$ imply

$$(\gamma - \tilde{\gamma}) F(x) = b[a(x) - \tilde{\gamma} x]. \tag{31}$$

Equations (18) and (31) yield

$$\left(\exp \int_{r_0}^{x} \frac{a'(t)}{\gamma t}\, dt\right)\Big/(a(x) - \tilde{\gamma} x) = \frac{b}{(\gamma - \tilde{\gamma}) F(r_0)}\,. \tag{32}$$

Differentiating (32) yields the homogeneous differential equation

$$a'(x)[a(x) - (\gamma + \tilde{\gamma})x] = -\gamma\tilde{\gamma} x. \tag{33}$$

Solving (33), we get

$$| a(x) - \gamma x |^{\tilde{\gamma}} \, | a(x) - \tilde{\gamma} x |^{-\gamma} = c. \tag{34}$$

By scaling $a(x)$, we can assume $c = 1$. Equation (34) is then symmetric in γ and $\tilde{\gamma}$, so we can assume $\tilde{\gamma} < \gamma$. (The case $\tilde{\gamma} > \gamma$ just reverses the role of f and $\tilde{f}$.) Let $\gamma = (1 + 1/\beta)^{-\beta}$, $\tilde{\gamma} = (1 + 1/\beta)^{-\beta-1}$. Then (34) becomes

$$| a(x) - (1 + 1/\beta)^{-\beta} x |^{\beta} = | a(x) - (1 + 1/\beta)^{-\beta-1} x |^{\beta+1}. \tag{35}$$

An easy argument eliminates the case $a(x) < (1 + 1/\beta)^{-\beta-1}x$, and so the absolute-value sign can be removed from the right-hand side of (35). The solutions of the resulting equation are then precisely the functions (27). Since f is a $(1 + 1/\beta)^{-\beta}$-semiwombat with average-value function $a_{\beta,i}$ ($i = 1$ or 2), it follows that $f = f_{\beta,i}$ up to a multiplicative constant.

4. Analyticity

We now head for another uniqueness theorem, one which asserts that we have already found all semiwombat functions which are analytic at $x = 0$.

Theorem 23. *If $a(x)$ is any real-valued solution of $(*)$ and, in addition, $a(0) = 0$, $a'(0) \geqslant 0$, and $a(x)$ is analytic at $x = 0$, then one of the following two statements holds*

(i) *there exist $\delta > 0$ and a real number $\lambda \geqslant 0$ such that $a(x) = \lambda x$ for $0 \leqslant x < \delta$.*

(ii) *there exist $\delta > 0$, a positive integer n, a real number $k > 0$, and $i = 1$ or 2 such that $a(x) = (1/k)\, a_{\beta,i}(kx)$ for $0 \leqslant x < \delta$, where $\beta = 1/n$.*

Proof. We begin by observing that the functions $a_{\beta,i}(x)$, $\beta = 1/n$, $i = 1, 2$, *are* analytic at $x = 0$. To see this, note that if $\beta = 1/n$ and $i = 1$ then $x = g_{\beta,1}(y) = (1 - 1/y)^{1/n}(1 + 1/ny)$, so $x^n = P_{n,1}(1/y)$, where $P_{n,1}(z) = (1 - z)(1 + z/n)^n$. Since $P'_{n,1}(1) \neq 0$, the function $P_{n,1}^{-1}$ with $P_{n,1}^{-1}(0) = 1$ is well defined and analytic on a complex neighborhood of zero. Since $y = f_{\beta,1}(x) = 1/P_{n,1}^{-1}(x^n)$, the analyticity of $f_{\beta,1}(x)$ at $x = 0$ is apparent. A similar argument holds for $f_{\beta,2}(x)$. The analyticity of $a_{\beta,i}(x)$ at $x = 0$ follows from (27).

Now let $a(x) = \sum_{j=1}^{\infty} b_j x^j$ for x small. By directly substituting this power series into $(*)$, we see that $2b_2 b_1 = b_2$ and, for each $j \geqslant 2$, there is a fixed polynomial Q_j in j variables such that

$$Q_j(b_1, b_2, \ldots, b_j) + (j + 1)\, b_{j+1} b_1^{\,j} = b_{j+1}$$

and $Q_j(\cdot, 0, 0, \ldots, 0) \equiv 0$. This limits the b_j's to the following possibilities:

(A) b_1 arbitrary, $b_j = 0$ for all $j \geqslant 2$.

(B) Let b_{n+1} be the *first* of the coefficients $b_2, b_3, \ldots$ to be different from zero. Then $b_1 = (n + 1)^{-1/n}$ and all later coefficients are uniquely determined by the value of b_{n+1}.

Case (A) clearly leads to statement (i). To show that case (B) implies (ii), we must first examine the power series expansions for $a_{\beta,i}(x)$, $\beta = 1/n$, $i = 1, 2$. We saw above that $y = f_{\beta,1}(x)$ is analytic at $x = 0$, with $1 - (1/y) = x^n/(1 + 1/ny)^n$. So $1 - 1/y = x^n A(x)$, where $A(x)$ is analytic at $x = 0$ and $A(0) = (1 + 1/n)^{-n}$. Then $y = (1 - x^n A(x))^{-1} = 1 + x^n A(x) + x^{2n}[A(x)]^2 + \cdots$ for x small. It is now clear that

$$f_{\beta,1}(x) = 1 + (1 + 1/n)^{-n} x^n + \cdots.$$

From (28), $a'_{\beta,1}(x) = (n + 1)^{-1/n}[1 + (1 + 1/n)^{-(n+1)} x^n + \cdots]$ and

$$a_{\beta,1}(x) = (n + 1)^{-1/n} x + c_{n+1,1} x^{n+1} + \cdots,\ c_{n+1,1} > 0 \qquad (\beta = 1/n).$$

In analogous fashion,

$$a_{\beta,2}(x) = (n+1)^{-1/n}x + c_{n+1,2}x^{n+1} + \cdots,\ c_{n+1,2} < 0 \qquad (\beta = 1/n).$$

We now pick $k > 0$ and $i = 1$ or 2 such that $k^n c_{n+1,i} = b_{n+1}$. Then $a(x)$ and $(1/k)\, a_{\beta,i}(kx)$, $\beta = 1/n$, both satisfy the hypotheses of the theorem and their power series agree through the first $(n+1)$ terms. It follows from (B) that $a(x) \equiv (1/k)\, a_{\beta,i}(kx)$ for x small, and so (ii) holds.

THEOREM 24. *If $a(x)$ is any complex-valued solution of $(*)$ with $a(0) = 0$ and $a(x)$ analytic at $x = 0$, then one of the following two statements holds:*

(i) *There exist $\delta > 0$ and a complex number λ such that $a(x) = \lambda x$ for $0 \leqslant x < \delta$.*

(ii) *There exist $\delta > 0$, a positive integer n, a complex number $k \neq 0$, an nth root of unity ω, and $i = 1$ or 2 such that $a(x) = (\omega/k)\, a_{\beta,i}(kx)$ for $0 \leqslant x < \delta$, where $\beta = 1/n$.*

Proof. If we again let $a(x) = \sum b_j x^j$ for x small and let b_{n+1} be as before, we again get $b_1{}^n = (n+1)^{-1}$, so $b_1 = \omega(n+1)^{-1/n}$, where $\omega^n = 1$. We next observe that, for $\beta = 1/n$ and $\omega^n = 1$, the functions $a_{\beta,i}(x)$ and $(1/\omega)\, a_{\beta,i}(\omega x)$ both satisfy the hypotheses of the theorem and their power series agree through the first $(n+1)$ terms, so $a_{\beta,i}(x) \equiv (1/\omega)\, a_{\beta,i}(\omega x)$ for (at least) x small. We claim that the function $H_{\beta,i}(x) = \omega a_{\beta,i}(x)$ satisfies $(*)$ for x small. For $H'_{\beta,i}(H_{\beta,i}(x)) = \omega a'_{\beta,i}(\omega a_{\beta,i}(x)) = \omega a'_{\beta,i}(a_{\beta,i}(\omega x)) = \omega a_{\beta,i}(\omega x)/\omega x = \omega a_{\beta,i}(x)/x = H_{\beta,i}(x)/x$. We now arbitrarily fix i and pick k such that $\omega k^n c_{n+1,i} = b_{n+1}$. Then, reasoning as before, $a(x) \equiv (\omega/k)\, a_{\beta,i}(kx)$ for x small, and (ii) is proved.

Remark. The interested reader may show that if $a(x)$ is a *real*-valued function satisfying the hypotheses of Theorem 24, then, for x small, either $a(x) = \lambda x$ for some real λ or $a(x) = (\omega/k)\, a_{\beta,i}(kx)$, where $\beta = 1/n$, $k > 0$, and $\omega = 1$ for n odd and $\omega = \pm 1$ for n even.

Remark. In general, the quantity δ of Theorems 23 and 24 can be smaller than the quantity r of $(*)$. For example, let

$$\begin{aligned} a(x) &= 0 \qquad \text{for} \quad 0 \leqslant x \leqslant 1 \\ &= 2x \qquad \text{for} \quad x > 1. \end{aligned}$$

Then $(*)$ holds on $(0, \infty)$, but clearly $\delta \leqslant 1$. We see in Lemma 26 that this cannot happen for "nice" functions $a(x)$.

LEMMA 25. *If the α-semiwombat function $f(x)$ is analytic at $x = x_0$, then $a(x)$ is analytic at $x = x_0$.*

Proof. First assume $x_0 > 0$. Since $F(x) = F(x_0) + \int_{x_0}^{x} f(t)\,dt$ and the Taylor series expansion for $f(t)$ around $t = x_0$ can be integrated term by term, it follows that $F(x)$ is analytic at $x = x_0$. Lemmas 7 and 9 now imply that $a'(x)$ is analytic at $x = x_0$, and then so is $a(x)$. If $x_0 = 0$, then $f(x) = cx^n[1 + \sum_{j=1}^{\infty} b_j x^j]$ and $F(x) = (c/(n+1))\,x^{n+1}[1 + \sum_{j=1}^{\infty} c_j x^j]$ for x small. Lemma 9 implies $\lim_{x\to 0} a'(x) = \lim_{x\to 0}(\alpha x f(x)/F(x)) = \alpha(n+1)$. By the mean-value theorem, $a'(0)$ exists and equals $\alpha(n+1)$. If we use the above power series to continue $f(x)$ and $F(x)$ off the real axis and then apply Lemma 9, we see that $a'(x)$ can be continued to a function $a'(z)$ analytic on (at least) a punctured complex neighborhood of zero. Moreover, $\lim_{z\to 0} a'(z) = \lim_{z\to 0}(\alpha z f(z)/F(z)) = \alpha(n+1) = a'(0)$, so $a'(z)$ is continuous at $z = 0$. Morera's theorem now gives the analyticity of $a'(z)$ at $z = 0$. Since $a(x) = \int_0^x a'(t)\,dt$, our lemma is proved.

LEMMA 26. *Suppose f_1 and f_2 are α-semiwombat functions on $[0, r_1)$ and $[0, r_2)$, respectively, with f_2 analytic at each point of $(0, r_2)$. If $f_1 = f_2$ on a neighborhood of zero, then $f_1 \equiv f_2$ on $[0, \min(r_1, r_2))$.*

Remark. The author does not know if the lemma is still true without the analyticity requirement. Furthermore, every semiwombat function known to the author is analytic except (possibly) at zero. So the lemma is very crude, but it suffices for our purposes.

Proof. If $f_1 \not\equiv f_2$, there exists a largest x_0 such that $f_1 = f_2$ on $[0, x_0]$, with $0 < x_0 < \min(r_1, r_2)$. Let $a_1 = a_2 = a$ on $[0, x_0]$. Let $\{x_n\}$ be a sequence strictly decreasing to x_0 with $a_1(x_n) \neq a_2(x_n)$. Since a_1 and a_2 are strictly increasing (by Lemmas 8 and 9) and continuous, $a_1(x_n) \downarrow a(x_0)$ and $a_2(x_n) \downarrow a(x_0)$. Since $0 < a(x_0) < x_0$ and a_1 and a_2 satisfy $(*)$, we see that, for n large, the equation $a'(w)/w = 1/x_n$ has (at least) two distinct roots $w = a_1(x_n)$ and $w = a_2(x_n)$. By Lemma 25, a is analytic on $(0, x_0)$, and so we can certainly apply Rolle's theorem to the function $H(w) = a'(w)/w$ on the interval $[a_1(x_n), a_2(x_n)]$ to conclude $H'(w_n) = 0$ for some $w_n \in (a_1(x_n), a_2(x_n))$. But $w_n > a(x_0)$ and $w_n \to a(x_0)$; hence the zeros of $H'(w)$ cluster at $a(x_0)$. By analyticity, $H'(w) \equiv 0$ and $a'(w)/w \equiv 1/x_0$ on $(0, x_0)$. But this contradicts the fact that we can solve $a'(w)/w = 1/x_n$ near $a(x_0)$, and proves the lemma.

THEOREM 27. *Suppose $f(x)$ is a γ-semiwombat function on $[0, r)$ which is analytic at $x = 0$. Then one of the following two statements holds:*

(i) *There exist a positive integer n and a real number $c \neq 0$ such that $f(x) = cx^n$ on $[0, r)$.*

(ii) *There exist a positive integer n, real numbers $c \neq 0$ and $k > 0$, and $i = 1$ or 2 such that $f(x) = cf_{\beta,i}(kx)$ or $f(x) = c\tilde{f}_{\beta,i}(kx)$ on $[0, r)$, where $\beta = 1/n$. (If $r = \infty$, then $i = 2$. If r is finite and $i = 1$, then $k \leqslant 1/r$.)*

Proof. By Lemma 25, $a(x)$ is analytic at $x = 0$. By Theorem 23, $a(x) = \lambda x$ or $a(x) = (1/k)\, a_{\beta,i}(kx)$, $\beta = 1/n$, for x small. The former case leads to $f(x) = cx^{\nu}$, $\nu > 0$, for x small (see the discussion preceding Theorem 16). By analyticity, ν must be a positive integer, and so (i) holds for x small. The latter case leads to $f(x) = cf_{\beta,i}(kx)$ or $f(x) = c\tilde{f}_{\beta,i}(kx)$ for x small. In either case, Lemma 26 is applicable, and the theorem follows.

COROLLARY 28. *There exists a γ-semiwombat function which is analytic at $x = 0$ if and only if $\gamma = (n+1)^{-1/n}$ or $\gamma = (n+1)^{-1-1/n}$ for some positive integer n.*

Proof. Immediate from Theorems 16, 17, 20, and 27.

5. Open Questions

Of course, the basic question which the author has not been able to answer is whether there are any semiwombat functions beyond those already found. Of particular interest is whether or not there exists an e^{-1}-semiwombat function, since we have exhibited α-semiwombats for every other α.

Of equal importance is the (possibly equivalent) question of whether we have found all solutions of $(*)$ with $0 < a(x) < x$ and $a \in C^1((0, r))$. Are all such solutions analytic on $(0, r)$? Are all solutions continuously differentiable at $x = 0$? (For $\beta > 1$, $a_{\beta,i}$ is not *twice* differentiable at $x = 0$.) Do they satisfy the hypothesis of Theorem 15? Can two solutions pass through the same point (other than the origin) with the same slope? Is the condition $a \in C^1((0, r))$ superfluous? Finally, can some sort of iterative scheme be set up using Eq. (16) and Schröder's equation (17) to generate solutions of $(*)$? It is hoped that this paper will generate enough interest that the author and others will be encouraged to pursue these matters.

References

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